MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
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CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Quadrature-based balanced truncation for Quadratic-bilinear systems

Reetish Padhi

joint work with Ion Victor Gosea at Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany

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Outline

- Part I: Linear time-invariant systems
  1. Definition and properties
  2. Balanced truncation (BT)
  3. Data-driven balancing (Quad BT)

- Part II: Quadratic-bilinear time-invariant systems
  1. Definition and properties
  2. Balanced truncation (BT)
  3. Data-driven balancing (Quad BT)

- Conclusion and outlook

Part I: Linear systems theory

LTI definition and kernels

- A linear time-invariant (LTI) system $\mathbf{\Sigma} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a dynamical system given by the following set of differential equations:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{y}(t) \in \mathbb{R}$ and $\mathbf{u}(t) \in \mathbb{R}$.

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- Solving for $\mathbf{y}(t)$ while assuming $\mathbf{x}(0) = 0$:

$$\mathbf{y}(t) = \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau = \int_0^t \underbrace{\mathbf{C}e^{\mathbf{A}\tau}\mathbf{B}}_{\mathbf{h}(\tau)}\mathbf{u}(t-\tau)d\tau = \underbrace{(\mathbf{h} * \mathbf{u})}_{\text{convolution}}(t)$$

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- The **impulse response** or **kernel** $\mathbf{h}(\tau) = \mathbf{C}e^{\mathbf{A}\tau}\mathbf{B}$ describes the input-output transfer in the time domain.

Gramians and Lyapunov equations

Definition

The infinite Gramians for a stable LTI system $\mathbf{\Sigma} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ *are defined as,*

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top t} dt, \tag{2}$$

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^\top t}\mathbf{C}^\top \mathbf{C} e^{\mathbf{A}t} dt. \tag{3}$$

- The reachability Gramian $\mathbf{P}$ quantifies how easily a state is reachable from the zero state.
- Similarly, the observability Gramian $\mathbf{Q}$ quantifies how easily a state is distinguishable from the zero state.

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- Similarly, the observability Gramian $\mathbf{Q}$ quantifies how easily a state is distinguishable from the zero state.
- The Gramians $\mathbf{P}, \mathbf{Q}$ satisfy the following (linear) Lyapunov matrix equations,

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top = \mathbf{0},$$

$$\mathbf{A}^\top \mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^\top \mathbf{C} = \mathbf{0}.$$

Balanced Truncation

- Originally introduced in:

Principal component analysis in linear systems: Controllability, observability, and model reduction, B. Moore, IEEE Transactions on Automatic Control, 26 (1), 1981.

and

Model reduction via balanced state space representations , L. Pernebo, L. Silverman, IEEE Transactions on Automatic Control, 27 (2), 382-387, 1982.

Square-root BT algorithm (LTI systems)

- Let $\mathbf{U}, \mathbf{L}$ be square factors of the Gramians, i.e., $\mathbf{P} = \mathbf{U}\mathbf{U}^T$ and $\mathbf{Q} = \mathbf{L}\mathbf{L}^T$.

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- Compute the truncated **SVD** of matrix $\mathbf{L}^T\mathbf{U}$:

$$\mathbf{L}^T\mathbf{U} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 \in \mathbb{R}^{r\times r} & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1^\top \\ \mathbf{Y}_2^\top \end{bmatrix}$$

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- Construct the model reduction bases

$$\mathbf{W}_r = \mathbf{L}\mathbf{Z}_1\mathbf{S}_1^{-1/2} \text{ and } \mathbf{V}_r = \mathbf{U}\mathbf{Y}_1\mathbf{S}_1^{-1/2}.$$

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- The matrices of reduced-order balanced system $\mathbf{\Sigma}_r$ are given by

$$\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{A}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},$$

$$\mathbf{B}_r = \mathbf{W}_r^T\mathbf{B} = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{B}\right),$$

$$\mathbf{C}_r = \mathbf{C}\mathbf{V}_r = \left(\mathbf{C}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2}.$$

Data-driven balancing (QuadBT)

- Originally introduced in:

Data-driven balancing of linear dynamical systems, I.V. Gosea, S. Gugercin, C. Beattie, SIAM Journal on Scientific Computing 44 (1), A554-A582, 2022.

Motivation

- In many applications, we might not have explicit access to the system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$!

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- **QuadBT** algorithm tries to address this very scenario by approximating the terms $\mathbf{L}^\top \mathbf{U}, \mathbf{L}^\top \mathbf{A} \mathbf{U}, \mathbf{L}^\top \mathbf{B}$ and $\mathbf{C} \mathbf{U}$ (**appearing in orange**) in the previous slide with data-driven quantities.

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- In many applications, we might not have explicit access to the system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$!
- **QuadBT** algorithm tries to address this very scenario by approximating the terms $\mathbf{L}^\top\mathbf{U}, \mathbf{L}^\top\mathbf{A}\mathbf{U}, \mathbf{L}^\top\mathbf{B}$ and $\mathbf{C}\mathbf{U}$ (**appearing in orange**) in the previous slide with data-driven quantities.
- **QuadBT** is different from other data-driven **BT** methods in that it is based solely on samples of the **kernels** (time domain) or **transfer function** (frequency domain) without requiring access to state trajectories or other state-based quantities.

Quadrature approximation of Gramians

- We use quadrature nodes to approximate the square root factors of the Gramians $\mathbf{L}$ and $\mathbf{U}$ as follows:

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top t}dt \approx \widetilde{\mathcal{P}} = \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top \mu_i} = \widetilde{\mathbf{U}}\widetilde{\mathbf{U}}^\top,$$

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^\top t}\mathbf{C}^\top\mathbf{C}e^{\mathbf{A}t}dt \approx \widetilde{\mathcal{Q}} = \sum_{j=1}^{N_q} \phi_j^2 e^{\mathbf{A}^\top \omega_j}\mathbf{C}^\top\mathbf{C}e^{\mathbf{A}\omega_j} = \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}^\top,$$

where N_p, N_q denote the number of quadrature nodes used in $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$ respectively, $\{\mu_i, \omega_j\}$ and $\{\rho_i, \phi_j\}$ denote the quadrature nodes and weights.

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- Observing $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$, we construct the quadrature based square factors $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{L}}$.

$$\mathbf{U} \approx \widetilde{\mathbf{U}} = \left[\rho_1 e^{\mathbf{A}\mu_1}\mathbf{B} \cdots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}}\mathbf{B}\right] \tag{4}$$

$$\mathbf{L} \approx \widetilde{\mathbf{L}} = \left[\phi_1 e^{\mathbf{A}^\top \omega_1}\mathbf{C}^\top \cdots \phi_{N_q} e^{\mathbf{A}^\top \omega_{N_q}}\mathbf{C}^\top\right]. \tag{5}$$

Replacing intrusive terms with data

- Recall the terms appearing in the **BT** algorithm i.e. $\mathbf{L}^\top\mathbf{U}$,$\mathbf{L}^\top\mathbf{A}\mathbf{U}$,$\mathbf{L}^\top\mathbf{B}$ and $\mathbf{C}\mathbf{U}$. Start by replacing the $\mathbf{L}^\top\mathbf{U}$ term with $\widetilde{\mathbb{H}} = \widetilde{\mathbf{L}}^\top\widetilde{\mathbf{U}}$.

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- The $\widetilde{\mathbb{H}}$ matrix can be expressed as samples of the kernel $\mathbf{h}(t)$ as shown below,

$$\widetilde{\mathbb{H}} = \begin{bmatrix} \rho_1\phi_1\mathbf{h}(\mu_1+\omega_1) & \dots & \rho_{N_p}\phi_1\mathbf{h}(\mu_{N_p}+\omega_1) \\ \vdots & \ddots & \vdots \\ \rho_1\phi_{N_q}\mathbf{h}(\mu_1+\omega_{N_q}) & \dots & \rho_{N_p}\phi_{N_q}\mathbf{h}(\mu_{N_p}+\omega_{N_q}) \end{bmatrix}$$

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- Similarly, for $\mathbf{L}^\top\mathbf{A}\mathbf{U}$ we have the term $\widetilde{\mathbb{M}} = \widetilde{\mathbf{L}}^\top\mathbf{A}\widetilde{\mathbf{U}}$ which can be written as

$$\widetilde{\mathbb{M}} = \begin{bmatrix} \rho_1\phi_1\mathbf{h}'(\mu_1+\omega_1) & \dots & \rho_{N_p}\phi_1\mathbf{h}'(\mu_{N_p}+\omega_1) \\ \vdots & \ddots & \vdots \\ \rho_1\phi_{N_q}\mathbf{h}'(\mu_1+\omega_{N_q}) & \dots & \rho_{N_p}\phi_{N_q}\mathbf{h}'(\mu_{N_p}+\omega_{N_q}) \end{bmatrix}$$

Replacing intrusive terms with data

- Similarly, $\mathbf{L}^\top \mathbf{B} \approx \widetilde{\mathbb{g}} = \widetilde{\mathbf{L}}^\top \mathbf{B}$ can be written as

$$\widetilde{\mathbb{g}} = \begin{bmatrix} \phi_1 \mathbf{h}(\omega_1) \\ \vdots \\ \phi_1 \mathbf{h}(\omega_{N_q}) \end{bmatrix}$$

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- Finally $\widetilde{\mathbb{h}} = \mathbf{C}\widetilde{\mathbf{U}}$ vector can also be expressed as samples of the kernel $\mathbf{h}(t)$ as shown below,

$$\widetilde{\mathbb{h}} = \begin{bmatrix} \rho_1 \mathbf{h}(\mu_1) & \dots & \rho_{N_p} \mathbf{h}(\mu_{N_p}) \end{bmatrix}$$

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- Having replaced all four terms with data-driven quantities, we are now ready to state the **QuadBT** algorithm.

- Put together the data matrices:

$$\widetilde{\mathbb{H}}, \quad \widetilde{\mathbb{M}}, \quad \widetilde{\mathbb{h}}, \quad \widetilde{\mathbb{g}}.$$

- Compute the truncated **SVD** of matrix $\widetilde{\mathbb{H}} \in \mathbb{R}^{N_q \times N_p}$:

$$\widetilde{\mathbb{H}} = \begin{bmatrix} \widetilde{\mathbf{Z}}_1 & \widetilde{\mathbf{Z}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{S}}_1 & \\ & \widetilde{\mathbf{S}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Y}}_1^\top \\ \widetilde{\mathbf{Y}}_2^\top \end{bmatrix}$$

- The matrices of the reduced-order data-driven system $\mathbf{\Sigma}_r$ are given by

$$\tilde{\mathbf{A}}_r = \cancel{\mathbf{W}_r^T \mathbf{A} \mathbf{V}_r} = \widetilde{\mathbf{S}}_1^{-1/2} \widetilde{\mathbf{Z}}_1^* \left(\widetilde{\mathbb{M}}\right) \widetilde{\mathbf{Y}}_1 \widetilde{\mathbf{S}}_1^{-1/2},$$

$$\tilde{\mathbf{B}}_r = \cancel{\mathbf{W}_r^T \mathbf{B}} = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^* \left(\widetilde{\mathbb{g}}\right),$$

$$\tilde{\mathbf{C}}_r = \cancel{\mathbf{C} \mathbf{V}_r} = \left(\widetilde{\mathbb{h}}\right) \widetilde{\mathbf{Y}}_1 \widetilde{\mathbf{S}}_1^{-1/2}.$$

Part II: Quadratic-bilinear (QB) systems

Definition and properties

- Quadratic-bilinear (QB) systems are denoted by the following equations,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{H}(\mathbf{x}(t)\otimes\mathbf{x}(t)) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

  where $\mathbf{u}(t)\in\mathbb{R}, \mathbf{x}(t)\in\mathbb{R}^n$ and $\mathbf{y}(t)\in\mathbb{R}$.
- We stick to the SISO case for simplicity.
- WLOG the matrix $\mathbf{H}$ can be considered to be the 1-matricization of a tensor $\mathcal{H}$, where $\mathcal{H}^{(2)} = \mathcal{H}^{(3)}$.
- A large class of nonlinear systems can be written as QB systems using lifting making it an important class of systems.

Kernels of QB systems

- The solution $\mathbf{x}(t)$ of the QB system $\boldsymbol{\Sigma}^{\mathsf{QB}}$ can be expressed as the sum of infinite subsystems as $\mathbf{x}(t) = \sum_{k=1}^{\infty} \mathbf{x}_k(t)$ given by

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) + \mathbf{B}\mathbf{u}(t),$$

$$\dot{\mathbf{x}}_k(t) = \mathbf{A}\mathbf{x}_k(t) + \mathbf{N}\mathbf{x}_{k-1}(t)\mathbf{u}(t) + \sum_{i=1}^{k-1} \mathbf{H}(\mathbf{x}_i(t) \otimes \mathbf{x}_{k-i}(t)) \text{ for } \; k > 1.$$

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- The solution of $\mathbf{x}_k(t)$ is given by

$$\mathbf{x}_k(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau + \sum_{i=1}^{k-1} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{H}(\mathbf{x}_i(\tau) \otimes \mathbf{x}_{k-i}(\tau))d\tau$$

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- Finally this gives us, $\mathbf{y}(t) = \mathbf{C}\sum_{k=1}^{\infty} \mathbf{x}_k(t)$.

Kernels of QB systems

$$\mathbf{y}(t) = \sum_{k=1}^{\infty}\left(\int_0^t \underbrace{\mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{N}\mathbf{x}_{k-1}(\tau)}_{\text{Bilinear contribution}}\mathbf{u}(\tau)d\tau + \sum_{i=1}^{k-1}\int_0^t \underbrace{\mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{H}(\mathbf{x}_i(\tau)\otimes\mathbf{x}_{k-i}(\tau))}_{\text{Quadratic contribution}}d\tau\right)$$

- Unlike the linear case, we don't have a single kernel $\mathbf{h}$ but a collection of them.
- Then, the QB kernels are,

$$\begin{aligned}
\mathbf{K}_1 &= \{\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\}\\
\mathbf{K}_2 &= \{\mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{B}, \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}(e^{\mathbf{A}t_2}\mathbf{B}\otimes e^{\mathbf{A}t_3}\mathbf{B})\}\\
\mathbf{K}_3 &= \{\mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{N}e^{\mathbf{A}t_3}\mathbf{B}, \mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{H}(e^{\mathbf{A}t_3}\mathbf{B}\otimes e^{\mathbf{A}t_4}\mathbf{B}),\\
&\quad \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}(e^{\mathbf{A}t_2}\mathbf{B}\otimes e^{\mathbf{A}t_3}\mathbf{H}(e^{\mathbf{A}t_4}\mathbf{B}\otimes e^{\mathbf{A}t_5}\mathbf{B}),\\
&\quad \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}(e^{\mathbf{A}t_2}\mathbf{B}\otimes e^{\mathbf{A}t_3}\mathbf{N}e^{\mathbf{A}t_4}\mathbf{B})), \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}(e^{\mathbf{A}t_2}\mathbf{N}e^{\mathbf{A}t_3}\mathbf{B}\otimes e^{\mathbf{A}t_4}\mathbf{B}),\\
&\quad \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}(e^{\mathbf{A}t_2}\mathbf{H}(e^{\mathbf{A}t_3}\mathbf{B}\otimes e^{\mathbf{A}t_4}\mathbf{B})\otimes e^{\mathbf{A}t_5}\mathbf{B}))\}\\
&\;\;\vdots
\end{aligned}$$

Kernel notation

- Having defined the kernels, we now need a way to notate them.
- Some examples:

$$\mathbf{h}_3^{\mathbf{N},\mathbf{N}}(t_1,t_2,t_3) = \mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{N}e^{\mathbf{A}t_3}\mathbf{B}$$

$$\mathbf{h}_3^{\mathbf{N},\mathbf{H}(-,-)}(t_1,t_2,t_3,t_4) = \mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{H}\left(e^{\mathbf{A}t_3}\mathbf{B}\otimes e^{\mathbf{A}t_4}\mathbf{B}\right)$$

$$\mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}(t_1,t_2,t_3,t_4) = \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}\left(e^{\mathbf{A}t_2}\mathbf{B}\otimes e^{\mathbf{A}t_3}\mathbf{N}e^{\mathbf{A}t_4}\mathbf{B}\right)$$

$$\mathbf{h}_4^{\mathbf{H}(\mathbf{H}(-,\mathbf{N}),-)}(t_1,\dots,t_6) = \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}\left(e^{\mathbf{A}t_2}\mathbf{H}\left(e^{\mathbf{A}t_3}\mathbf{B}\otimes e^{\mathbf{A}t_4}\mathbf{N}e^{\mathbf{A}t_5}\mathbf{B}\right)\otimes e^{\mathbf{A}t_6}\mathbf{B}\right)$$

Summary - QB kernels

- The output of the QB system can be broken into contributions from infinite subsystems.

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}_1(t) + \mathbf{C}\mathbf{x}_2(t) + \dots$$

- Each subsystem has associated with it some finite number of kernels.

$$\mathbf{K}_2 = \{\mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{B}, \mathbf{C}e^{\mathbf{A}t_1}\mathbf{H}(e^{\mathbf{A}t_2}\mathbf{B} \otimes e^{\mathbf{A}t_3}\mathbf{B})\}$$

- There is a way to enumerate and notate the kernels of the QB system.

$$\mathbf{h}_3^{\mathbf{N},\mathbf{N}}(t_1, t_2, t_3) = \mathbf{C}e^{\mathbf{A}t_1}\mathbf{N}e^{\mathbf{A}t_2}\mathbf{N}e^{\mathbf{A}t_3}\mathbf{B}$$

Balanced truncation (BT) for QB systems

Gramians for QB systems

- The QB Gramians are difficult to compute, which is why using truncated Gramians is sometimes advised [Benner/Goyal '17] .

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- The (truncated) Gramians of the QB system $\mathbf{P}_\tau, \mathbf{Q}_\tau$ are defined as,

$$\mathbf{P}_1 = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top t}dt \qquad \mathbf{Q}_1 = \int_0^\infty e^{\mathbf{A}^\top t}\mathbf{C}^\top \mathbf{C} e^{\mathbf{A}t}dt$$

$$\mathbf{P}_2 = \int_0^\infty e^{\mathbf{A}t}\mathbf{N}\mathbf{P}_1\mathbf{N}^\top e^{\mathbf{A}^\top t}dt \qquad \mathbf{Q}_2 = \int_0^\infty e^{\mathbf{A}^\top t}\mathbf{N}^\top \mathbf{P}_1 \mathbf{N} e^{\mathbf{A}t}dt$$

$$\mathbf{P}_3 = \int_0^\infty e^{\mathbf{A}t}\mathbf{H}(\mathbf{P}_1 \otimes \mathbf{P}_1)\mathbf{H}^\top e^{\mathbf{A}^\top t}dt \qquad \mathbf{Q}_3 = \int_0^\infty e^{\mathbf{A}^\top t}\mathbf{H}^{(2)}(\mathbf{P}_1 \otimes \mathbf{Q}_1)\mathbf{H}^{(2)\top} e^{\mathbf{A}t}dt$$

$$\mathbf{P}_\tau = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 \qquad \mathbf{Q}_\tau = \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3$$

Gramians for QB systems

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\begin{aligned}
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\mathbf{P}_2 &= \int_0^\infty e^{\mathbf{A}t}\mathbf{N}\mathbf{P}_1\mathbf{N}^\top e^{\mathbf{A}^\top t} dt & \mathbf{Q}_2 &= \int_0^\infty e^{\mathbf{A}^\top t}\mathbf{N}^\top \mathbf{P}_1 \mathbf{N} e^{\mathbf{A}t} dt \\
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\mathbf{P}_\tau &= \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 & \mathbf{Q}_\tau &= \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3
\end{aligned}
$$

- They present the balanced truncation for QB systems using the truncated Gramians.

Classical BT algorithm (QB systems)

- Let $\mathbf{U}, \mathbf{L}$ be square factors of $\mathbf{P}_\tau = \mathbf{U}\mathbf{U}^T$ and $\mathbf{Q}_\tau = \mathbf{L}\mathbf{L}^T$.
- Compute the truncated **SVD** of matrix $\mathbf{L}^T\mathbf{U}$:

$$\mathbf{L}^T\mathbf{U} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1^\top \\ \mathbf{Y}_2^\top \end{bmatrix}$$

- Construct the model reduction bases

$$\mathbf{W}_r = \mathbf{L}\mathbf{Z}_1\mathbf{S}_1^{-1/2} \text{ and } \mathbf{V}_r = \mathbf{U}\mathbf{Y}_1\mathbf{S}_1^{-1/2}$$

- The matrices of reduced-order balanced system $\mathbf{\Sigma}_r$ are given by

$$\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{A}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},$$

$$\mathbf{B}_r = \mathbf{W}_r^T\mathbf{B} = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{B}\right),$$

$$\mathbf{C}_r = \mathbf{C}\mathbf{V}_r = \left(\mathbf{C}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},$$

$$\mathbf{H}_r = \mathbf{W}_r^T\mathbf{H}(\mathbf{V}_r \otimes \mathbf{V}_r) = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{H}(\mathbf{U}\otimes\mathbf{U})\right)\left(\mathbf{Y}_1\mathbf{S}_1^{-1/2}\otimes\mathbf{Y}_1\mathbf{S}_1^{-1/2}\right),$$

$$\mathbf{N}_r = \mathbf{W}_r^T\mathbf{N}\mathbf{V}_r = \mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{N}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2}.$$

QuadBT for QB systems

- Just like in the linear and LQO case, the truncated QB Gramians and their square root factors are approximated as:

$$\widetilde{\mathcal{P}}_1 = \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top \mu_i} \qquad \widetilde{\mathbf{U}}_1 = \left[\rho_1 e^{\mathbf{A}\mu_1}\mathbf{B} \ldots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}}\mathbf{B}\right]$$

$$\widetilde{\mathcal{P}}_2 = \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i}\mathbf{N}\widetilde{\mathbf{U}}_1\widetilde{\mathbf{U}}_1^\top\mathbf{N}^\top e^{\mathbf{A}^\top \mu_i} \qquad \widetilde{\mathbf{U}}_2 = \left[\rho_1 e^{\mathbf{A}\mu_1}\mathbf{N}\widetilde{\mathbf{U}}_1 \ldots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}}\mathbf{N}\widetilde{\mathbf{U}}_1\right]$$

$$\widetilde{\mathcal{P}}_3 = \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i}\mathbf{H}\left(\widetilde{\mathcal{P}}_1 \otimes \widetilde{\mathcal{P}}_1\right)\mathbf{H}^\top e^{\mathbf{A}\mu_i} \qquad \widetilde{\mathbf{U}}_3 = \left[\rho_1 e^{\mathbf{A}\mu_1}\mathbf{H}\left(\widetilde{\mathbf{U}}_1 \otimes \widetilde{\mathbf{U}}_1\right) \ldots \right.$$

$$\left. \ldots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}}\mathbf{H}\left(\widetilde{\mathbf{U}}_1 \otimes \widetilde{\mathbf{U}}_1\right)\right]$$

- Finally we write $\widetilde{\mathbf{U}} = \left[\widetilde{\mathbf{U}}_1\widetilde{\mathbf{U}}_2\widetilde{\mathbf{U}}_3\right]$.
- A similar calculation follows for $\widetilde{\mathbf{L}}$.
- For simplicity, we assume unity weights and $N_p = N_q = N$.

Intrusive terms to be replaced!

Let $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{L}}$ be as discussed above, define the matrix $\widetilde{\mathbb{H}} = \widetilde{\mathbf{L}}^\top \widetilde{\mathbf{U}}$, then we have,

$$\widetilde{\mathbb{H}} = \begin{cases} \mathbf{h}_1(\omega_i + \mu_j), & \text{for } \widetilde{\mathbf{L}}_1^\top \widetilde{\mathbf{U}}_1, i,j = \overline{1,N} \\ \mathbf{h}_2^{\mathbf{N}}(\omega_i + \mu_j, \mu_k), & \text{for } \widetilde{\mathbf{L}}_1^\top \widetilde{\mathbf{U}}_2, i-k = \overline{1,N} \\ \mathbf{h}_2^{\mathbf{H}}(\omega_i + \mu_j, \mu_k, \mu_l), & \text{for } \widetilde{\mathbf{L}}_1^\top \widetilde{\mathbf{U}}_3, i-l = \overline{1,N} \\ \mathbf{h}_2^{\mathbf{N}}(\omega_i, \omega_j + \mu_k), & \text{for } \widetilde{\mathbf{L}}_2^\top \widetilde{\mathbf{U}}_1, i-k = \overline{1,N} \\ \mathbf{h}_3^{\mathbf{N},\mathbf{N}}(\omega_i, \omega_j + \mu_k, \mu_l), & \text{for } \widetilde{\mathbf{L}}_2^\top \widetilde{\mathbf{U}}_2, i-l = \overline{1,N} \\ \mathbf{h}_3^{\mathbf{N},\mathbf{H}}(\omega_i, \omega_j + \mu_k, \mu_l, \mu_m), & \text{for } \widetilde{\mathbf{L}}_2^\top \widetilde{\mathbf{U}}_3, i-m = \overline{1,N} \\ \mathbf{h}_2^{\mathbf{H}}(\omega_i, \mu_k, \omega_j + \mu_l)^\top, & \text{for } \widetilde{\mathbf{L}}_3^\top \widetilde{\mathbf{U}}_1, i-l = \overline{1,N} \\ \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}(\omega_i, \mu_k, \omega_j + \mu_l, \mu_m)^\top, & \text{for } \widetilde{\mathbf{L}}_3^\top \widetilde{\mathbf{U}}_2, i-m = \overline{1,N} \\ \mathbf{h}_3^{\mathbf{H}(-,\mathbf{H})}(\omega_i, \mu_k, \omega_j + \mu_l, \mu_m, \omega_n)^\top, & \text{for } \widetilde{\mathbf{L}}_3^\top \widetilde{\mathbf{U}}_3, i-n = \overline{1,N} \end{cases}$$

$\widetilde{\mathbb{H}}$ has the structure of a Hankel matrix (see [Antoulas '05]) given by

$$\widetilde{\mathbb{H}} = \begin{bmatrix} \widetilde{\mathbf{L}}_1^\top \widetilde{\mathbf{U}}_1 & \widetilde{\mathbf{L}}_1^\top \widetilde{\mathbf{U}}_2 & \widetilde{\mathbf{L}}_1^\top \widetilde{\mathbf{U}}_3 \\ \widetilde{\mathbf{L}}_2^\top \widetilde{\mathbf{U}}_1 & \widetilde{\mathbf{L}}_2^\top \widetilde{\mathbf{U}}_2 & \widetilde{\mathbf{L}}_2^\top \widetilde{\mathbf{U}}_3 \\ \widetilde{\mathbf{L}}_3^\top \widetilde{\mathbf{U}}_1 & \widetilde{\mathbf{L}}_3^\top \widetilde{\mathbf{U}}_2 & \widetilde{\mathbf{L}}_3^\top \widetilde{\mathbf{U}}_3 \end{bmatrix} \in \mathbb{R}^{(N+N^2+N^3)\times(N+N^2+N^3)}$$

It's all data!

- Construct the quadrature-based approximations:

$$\widetilde{\mathbb{M}} = \widetilde{\mathbf{L}}^T \mathbf{A} \widetilde{\mathbf{U}} \approx \mathbf{L}^\top \mathbf{A} \mathbf{U}$$

$$\widetilde{\mathbb{N}} = \widetilde{\mathbf{L}}^T \mathbf{N} \widetilde{\mathbf{U}} \approx \mathbf{L}^\top \mathbf{N} \mathbf{U}$$

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$$\widetilde{\mathbb{h}} = \widetilde{\mathbf{L}}^\top \mathbf{B} \approx \mathbf{L}^\top \mathbf{B}$$

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- The entries of each of these matrices $\widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{K}}, \widetilde{\mathbb{h}}$ and $\widetilde{\mathbb{g}}$ can be written in terms of the samples of QB kernels.

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- The entries of each of these matrices $\widetilde{\mathbb{M}}$, $\widetilde{\mathbb{N}}$, $\widetilde{\mathbb{K}}$, $\widetilde{\mathbb{h}}$ and $\widetilde{\mathbb{g}}$ can be written in terms of the samples of QB kernels.
- We are finally ready to present the Quad BT algorithm for QB systems.

QuadBT algorithm for QB systems

- Put together the data matrices:

$$\widetilde{\mathbb{H}},\quad \widetilde{\mathbb{K}},\quad \widetilde{\mathbb{N}},\quad \widetilde{\mathbb{M}},\quad \widetilde{\mathbb{h}},\quad \widetilde{\mathbb{g}}.$$

- Compute the truncated **SVD** of matrix $\widetilde{\mathbb{H}}$:

$$\widetilde{\mathbb{H}} = \begin{bmatrix} \widetilde{\mathbf{Z}}_1 & \widetilde{\mathbf{Z}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{S}}_1 & \\ & \widetilde{\mathbf{S}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Y}}_1^\top \\ \widetilde{\mathbf{Y}}_2^\top \end{bmatrix}.$$

- The matrices of the reduced-order data-driven system $\mathbf{\Sigma}_r$ are given by

$$\begin{aligned}
\tilde{\mathbf{A}}_r &= \cancel{\mathbf{W}_r^T \mathbf{A} \mathbf{V}_r} = \widetilde{\mathbf{S}}_1^{-1/2} \widetilde{\mathbf{Z}}_1^\top \left(\widetilde{\mathbb{M}}\right) \widetilde{\mathbf{Y}}_1 \widetilde{\mathbf{S}}_1^{-1/2},\\
\tilde{\mathbf{B}}_r &= \cancel{\mathbf{W}_r^T \mathbf{B}} = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top \left(\widetilde{\mathbb{g}}\right),\\
\tilde{\mathbf{C}}_r &= \cancel{\mathbf{C}\mathbf{V}_r} = \left(\widetilde{\mathbb{h}}\right) \widetilde{\mathbf{Y}}_1 \widetilde{\mathbf{S}}_1^{-1/2}.\\
\tilde{\mathbf{N}}_r &= \cancel{\mathbf{L}^\top \mathbf{N} \mathbf{U}} = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\widetilde{\mathbb{N}}) \widetilde{\mathbf{Y}}_1 \widetilde{\mathbf{S}}_1^{-1/2}\\
\tilde{\mathbf{H}}_r &= \cancel{\mathbf{L}^\top \mathbf{H}(\mathbf{U} \otimes \mathbf{U})} = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\widetilde{\mathbb{K}}) (\widetilde{\mathbf{Y}}_1 \widetilde{\mathbf{S}}_1^{-1/2} \otimes \widetilde{\mathbf{Y}}_1 \widetilde{\mathbf{S}}_1^{-1/2})
\end{aligned}$$

Are we done? Not quite.

1. How to avoid forming the Kronecker product $\widetilde{\mathbf{Y}}_1\widetilde{\mathbf{S}}_1^{-1/2} \otimes \widetilde{\mathbf{Y}}_1\widetilde{\mathbf{S}}_1^{-1/2}$ appearing in the expression for $\tilde{\mathbf{H}}_r$?

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   To deal with this, we consider replacing $\widetilde{\mathbf{U}}_1$ with a fewer node approximation $\widehat{\widetilde{\mathbf{U}}}_1$ in the expression for $\widetilde{\mathbf{U}}_2$ and $\widetilde{\mathbf{U}}_3$.

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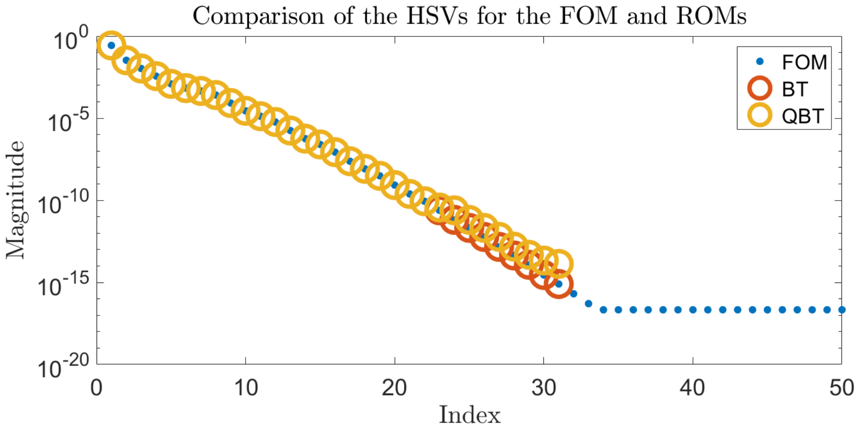
$$\widetilde{\mathbf{U}}_2 = \left[\rho_1 e^{\mathbf{A}\mu_1}\mathbf{N}\cancel{\widetilde{\mathbf{U}}_1} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}}\mathbf{N}\cancel{\widetilde{\mathbf{U}}_1}\right] \in \mathbb{R}^{n \times N_p^2}$$

$$\widehat{\widetilde{\mathbf{U}}}_2 = \left[\rho_1 e^{\mathbf{A}\mu_1}\mathbf{N}\widehat{\widetilde{\mathbf{U}}}_1 \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}}\mathbf{N}\widehat{\widetilde{\mathbf{U}}}_1\right] \in \mathbb{R}^{n \times N_p n_1}$$

   where $n_1 << N_p$.

Numerical results (Burgers Quadratic)

- FOM = Burgers quadratic equation with (n=150)
- Construct ROMs of dimension $r = 31$ using **BT** and **QuadBT** .



Conclusions and Outlook

- Successfully extended Quad BT to QB systems
- MIMO case follows immediately with the only difference being that the kernels are no longer scalars but matrices in $\mathbb{R}^{m \times p}$
- Adaptive schemes to pick quadrature nodes cleverly would be something interesting to try